

# ON THE STRONG CHROMATIC INDEX AND MAXIMUM INDUCED MATCHING OF TREE-COGRAPHS AND PERMUTATION GRAPHS

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**Abstract.** We show that there exist linear-time algorithms that compute the strong chromatic index and a maximum induced matching of tree-cographs when the decomposition tree is a part of the input. We also show that there exists an efficient algorithm for the strong chromatic index of permutation graphs.

## 1 Introduction

**Definition 1.** Let  $G = (V, E)$  be a graph. A strong edge coloring of  $G$  is a proper edge coloring such that no edge is adjacent to two edges of the same color.

Equivalently, a strong edge coloring of  $G$  is a vertex coloring of  $L(G)^2$ , the square of the linegraph of  $G$ . The strong chromatic index of  $G$  is the minimal integer  $k$  such that  $G$  has a strong edge coloring with  $k$  colors. We denote the strong chromatic index of  $G$  by  $sx'(G)$ .

The class of tree-cographs was introduced by Tinhofer in [16].

**Definition 2.** Tree-cographs are defined recursively by the following rules.

1. Every tree is a tree-cograph.
2. If  $G$  is a tree-cograph then also the complement  $\bar{G}$  of  $G$  is a tree-cograph.
3. For  $k \geq 2$ , if  $G_1, \dots, G_k$  are connected tree-cographs then also the disjoint union is a tree-cograph.

Let  $G$  be a tree-cograph. A decomposition tree for  $G$  consists of a rooted binary tree  $T$  in which each internal node, including the root, is labeled as a join node  $\otimes$  or a union node  $\oplus$ . The leaves of  $T$  are labeled by trees or complements of trees. It is easy to see that a decomposition tree for a tree-cograph  $G$  can be obtained in  $O(n^3)$  time.

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## 2 The strong chromatic index of tree-cographs

The linegraph  $L(G)$  of a graph  $G$  is the intersection graph of the edges of  $G$  [1]. It is well-known that, when  $G$  is a tree then the linegraph  $L(G)$  of  $G$  is a claw-free blockgraph [11]. A graph is chordal if it has no induced cycles of length more than three [7]. Notice that blockgraphs are chordal.

A vertex  $x$  in a graph  $G$  is simplicial if its neighborhood  $N(x)$  induces a clique in  $G$ . Chordal graphs are characterized by the property of having a perfect elimination ordering, which is an ordering  $[v_1, \dots, v_n]$  of the vertices of  $G$  such that  $v_i$  is simplicial in the graph induced by  $\{v_i, \dots, v_n\}$ . A perfect elimination ordering of a chordal graph can be computed in linear time [15]. This implies that chordal graphs have at most  $n$  maximal cliques, and the clique number can be computed in linear time.

**Theorem 1 ([2]).** *If  $G$  is a chordal graph then  $L(G)^2$  is also chordal.*

*Proof.* Any chordal graph is the intersection graph of a collection of subtrees of a tree. Let  $G$  be the intersection graph of a collection of subtrees of a tree. An intersection model for  $L(G)^2$  is obtained by taking the union of every pair of intersecting subtrees.  $\square$

**Theorem 2 ([4]).** *Let  $k \in \mathbb{N}$  and let  $k \geq 4$ . Let  $G$  be a graph and assume that  $G$  has no induced cycles of length at least  $k$ . Then  $L(G)^2$  has no induced cycles of length at least  $k$ .*

**Lemma 1.** *Tree-cographs have no induced cycles of length more than four.*

*Proof.* Let  $G$  be a tree-cograph.

First observe that trees are bipartite. It follows that complements of trees have no induced cycles of length more than four.

We prove the claim by induction on the depth of a decomposition tree for  $G$ . If  $G$  is the union of two tree-cographs  $G_1$  and  $G_2$  then the claim follows by induction since any induced cycle is contained in one of  $G_1$  and  $G_2$ . Assume  $G$  is the join of two tree-cographs  $G_1$  and  $G_2$ . Assume that  $G$  has an induced cycle  $C$  of length at least five. We may assume that  $C$  has at least one vertex in each of  $G_1$  and  $G_2$ . If one of  $G_1$  and  $G_2$  has more than two vertices of  $C$ , then  $C$  has a vertex of degree at least three, which is a contradiction.  $\square$

**Lemma 2.** *Let  $T$  be a tree. Then  $L(\bar{T})^2$  is a clique.*

*Proof.* Consider two non-edges  $\{a, b\}$  and  $\{p, q\}$  of  $T$ . If the non-edges share an endpoint then they are adjacent in  $L(\bar{T})^2$  since they are already adjacent in  $L(\bar{T})$ . Otherwise, since  $T$  is a tree, at least one pair of  $\{a, p\}$ ,  $\{a, q\}$ ,  $\{b, p\}$  and  $\{b, q\}$  is a non-edge in  $T$ , otherwise  $T$  has a 4-cycle. By definition,  $\{a, b\}$  and  $\{p, q\}$  are adjacent in  $L(\bar{T})^2$ .  $\square$

If  $G$  is the union of two tree-cographs  $G_1$  and  $G_2$  then the maximal cardinality of a clique in  $L(G)^2$  is, simply, the maximum over the clique numbers of  $L(G_1)^2$  and  $L(G_2)^2$ . The following lemma deals with the join of two tree-cographs.

**Lemma 3.** Let  $P$  and  $Q$  be tree-cographs and let  $G$  be the join of  $P$  and  $Q$ . Let  $X$  be the set of edges that have one endpoint in  $P$  and one endpoint in  $Q$ . Then

- (a)  $X$  forms a clique in  $L(G)^2$ ,
- (b) every edge of  $X$  is adjacent in  $L(G)^2$  to every edge of  $P$  and to every edge of  $Q$ ,  
and
- (c) every edge of  $P$  is adjacent in  $L(G)^2$  to every edge of  $Q$ .

*Proof.* This is an immediate consequence of the definitions.  $\square$

For  $k \geq 3$ , a  $k$ -sun is a graph which consists of a clique with  $k$  vertices and an independent set with  $k$  vertices. There exist orderings  $c_1, \dots, c_k$  and  $s_1, \dots, s_k$  of the vertices in the clique and independent set such that each  $s_i$  is adjacent to  $c_i$  and to  $c_{i+1}$  for  $i = 1, \dots, k-1$  and such that  $s_k$  is adjacent to  $c_k$  and  $c_1$ . A graph is strongly chordal if it is chordal and has no  $k$ -sun, for  $k \geq 3$  [8].

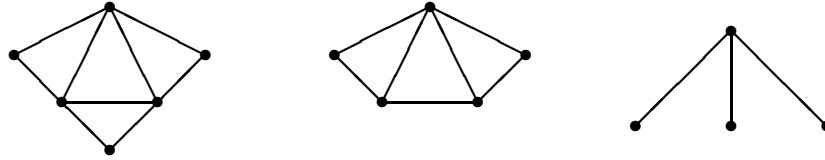


Fig. 1. A 3-sun, a gem and a claw

**Lemma 4.** Let  $T$  be a tree. Then  $L(T)^2$  is strongly chordal.

*Proof.* When  $T$  is a tree then  $L(T)$  is a blockgraph. Obviously, blockgraphs are strongly chordal. Lubiw proves in [14] that all powers of strongly chordal graphs are strongly chordal.  $\square$

We strengthen the result of Lemma 4 as follows. Ptolemaic graphs are graphs that are both distance hereditary and chordal [12]. Ptolemaic graphs are gem-free chordal graphs. The following theorem characterizes ptolemaic graphs.

**Theorem 3 ([12]).** A connected graph is ptolemaic if and only if for all pairs of maximal cliques  $C_1$  and  $C_2$  with  $C_1 \cap C_2 \neq \emptyset$ , the intersection  $C_1 \cap C_2$  separates  $C_1 \setminus C_2$  from  $C_2 \setminus C_1$ .

**Lemma 5.** Let  $T$  be a tree. Then  $L(T)^2$  is ptolemaic.

*Proof.* Consider  $L(T)$ . Let  $C$  be a block and let  $P$  and  $Q$  be two blocks that each intersect  $C$  in one vertex. Since  $L(T)$  is claw-free, the intersections of  $P \cap C$  and  $Q \cap C$  are distinct vertices. The intersection of the maximal cliques  $P \cup C$  and  $Q \cup C$ , which is  $C$ , separates  $P \setminus Q$  and  $Q \setminus P$  in  $L(T)^2$ . Since all intersecting pairs of maximal cliques are of this form, this proves the lemma.  $\square$

**Corollary 1.** *Let  $G$  be a tree-cograph. Then  $L(G)^2$  has a decomposition tree with internal nodes labeled as join nodes and union nodes and where the leaves are labeled as ptolemaic graphs.*

From Corollary 1 it follows that  $L(G)^2$  is perfect [5], that is,  $L(G)^2$  has no odd holes or odd antiholes [13]. This implies that the chromatic number of  $L(G)^2$  is equal to the clique number. Therefore, to compute the strong chromatic index of a tree-cograph  $G$  it suffices to compute the clique number of  $L(G)^2$ .

**Theorem 4.** *Let  $G$  be a tree-cograph and let  $T$  be a decomposition tree for  $G$ . There exists a linear-time algorithm that computes the strong chromatic index of  $G$ .*

*Proof.* First assume that  $G = (V, E)$  is a tree. Then the strong chromatic index of  $G$  is

$$s\chi'(G) = \max \{ d(x) + d(y) - 1 \mid (x, y) \in E \} \quad (1)$$

where  $d(x)$  is the degree of the vertex  $x$ . To see this notice that Formula (1) gives the clique number of  $L(G)^2$ .

Assume that  $G$  is the complement of a tree. By Lemma 2 the strong chromatic index is the number of nonedges in  $G$ , which is

$$s\chi'(G) = \binom{n}{2} - (n - 1).$$

Assume that  $G$  is the union of two tree-cographs  $G_1$  and  $G_2$ . Then, obviously,

$$s\chi'(G) = \max \{ s\chi'(G_1), s\chi'(G_2) \}.$$

Finally, assume that  $G$  is the join of two tree-cographs  $G_1$  and  $G_2$ . Let  $X$  be the set of edges of  $G$  that have one endpoint in  $G_1$  and the other in  $G_2$ . Then, by Lemma 3, we have

$$s\chi'(G) = |X| + s\chi'(G_1) + s\chi'(G_2).$$

The decomposition tree for  $G$  has  $O(n)$  nodes. For the trees the strong chromatic index can be computed in linear time. In all other cases, the evaluation of  $s\chi'(G)$  takes constant time. It follows that this algorithm runs in  $O(n)$  time, when a decomposition tree is a part of the input.  $\square$

### 3 Induced matching in tree-cographs

Consider a strong edge coloring of a graph  $G$ . Then each color class is an induced matching in  $G$ , which is an independent set in  $L(G)^2$  [2]. In this section we show that the maximal value of an induced matching in  $G$  can be computed in linear time. Again, we assume that a decomposition tree is a part of the input.

**Theorem 5.** *Let  $G$  be a tree-cograph and let  $T$  be a decomposition tree for  $G$ . Then the maximal number of edges in an induced matching in  $G$  can be computed in linear time.*

*Proof.* In this proof we denote the maximal cardinality of an induced matching in a graph  $G$  by  $iv(G)$ .

First assume that  $G$  is a tree. Since the maximum induced matching problem can be formulated in monadic second-order logic, there exists a linear-time algorithm to compute the maximal cardinality of an induced matching in  $G$ .

Assume that  $G$  is the complement of a tree. By lemma 2  $L(G)^2$  is a clique. Thus the maximal cardinality of an induced matching in  $G$  is one if  $G$  has a nonedge and otherwise it is zero.

Assume that  $G$  is the union of two tree-cographs  $G_1$  and  $G_2$ . Then

$$iv(G) = iv(G_1) + iv(G_2).$$

Assume that  $G$  is the join of two tree-cographs  $G_1$  and  $G_2$ . Then

$$iv(G) = \max \{ iv(G_1), iv(G_2), 1 \}.$$

This proves the theorem. □

## 4 Permutation graphs

A permutation diagram on  $n$  points is obtained as follows. Consider two horizontal lines  $L_1$  and  $L_2$  in the Euclidean plane. For each line  $L_i$  consider a linear ordering  $\prec_i$  of  $\{1, \dots, n\}$  and put points  $1, \dots, n$  on  $L_i$  in this order. For  $k = 1, \dots, n$  connect the two points with the label  $k$  by a straight line segment.

**Definition 3 ([10]).** *A graph  $G$  is a permutation graph if it is the intersection graph of the line segments in a permutation diagram.*

Consider two horizontal lines  $L_1$  and  $L_2$  and on each line  $L_i$  choose  $n$  intervals. Connect the left - and right endpoint of the  $k^{\text{th}}$  interval on  $L_1$  with the left - and right endpoint of the  $k^{\text{th}}$  interval on  $L_2$ . Thus we obtain a collection of  $n$  trapezoids. We call this a trapezoid diagram.

**Definition 4.** *A graph is a trapezoid graph if it is the intersection graph of a collection of trapezoids in a trapezoid diagram.*

**Lemma 6.** *If  $G$  is a permutation graph then  $L(G)^2$  is a trapezoid graph.*

*Proof.* Consider a permutation diagram for  $G$ . Each edge of  $G$  corresponds to two intersecting line segments in the diagram. The four endpoints of a pair of intersecting line segments define a trapezoid. Two vertices in  $L(G)^2$  are adjacent exactly when the corresponding trapezoids intersect (see Proposition 1 in [3]). □

**Theorem 6.** *There exists an  $O(n^4)$  algorithm that computes a strong edge coloring in permutation graphs.*

*Proof.* Dagan, *et al.*, show that a trapezoid graph can be colored by a greedy coloring algorithm. It is easy to see that this algorithm can be adapted so that it finds a strong edge-coloring in permutation graphs.  $\square$

*Remark 1.* A somewhat faster coloring algorithm for trapezoid graphs appears in [9]. Their algorithm runs in  $O(n \log n)$  time where  $n$  is the number of vertices in the trapezoid graph. An adaption of their algorithm yields a strong edge coloring for permutation graphs that runs in  $O(m \log n)$  time, where  $n$  and  $m$  are the number of vertices and edges in the permutation graph.

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